

# Exact solution of one class of Maryland model

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The Hamiltonian  $H$  of one-body Maryland model is defined as the sum of a linear unperturbed Hamiltonian  $H_0$  ( $H_{0nn} = n\omega$ ) and the interaction  $V$ , which is a Toeplitz matrix. Maryland model with a doubly infinite Hilbert space are exactly solved. Special cases of one-body Maryland model include the original Maryland model (Phys. Rev. Lett. 49, 833 (1982) and Physica 10D, 369 (1984)), which describes a quantum kicked linear rotator and single band Bloch oscillations. Maryland model and single band Bloch oscillations are the same Hamiltonian in two different representations. A special case of many-body Maryland model is Luttinger model.

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## I. EXACT SOLUTION

The Hamiltonian of one-body Maryland model is defined as

$$H = H_0 + V. \quad (1)$$

Both the unperturbed Hamiltonian  $H_0$  and the interaction  $V$  are doubly infinite matrices; the indices  $n$  and  $m$  of  $H_{0nm}$  and  $V_{nm}$  run from  $-\infty$  to  $\infty$ .  $H_0$  is a diagonal matrix; its diagonal matrix elements are  $\{-\infty, \dots, -2, -1, 0, 1, 2, \dots, \infty\} \times \omega(t)$ .  $V$  is a Toeplitz matrix:  $V_{nm} = V_{n-m}$ .

We want to solve Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(t) = H\psi(t). \quad (2)$$

The unitary operator  $U(t, 0)$  dictates the dynamics of the system.

$$\psi(t) = U(t, 0)\psi(0). \quad (3)$$

In the interaction picture,

$$i\hbar \frac{\partial}{\partial t} U_I(t, 0) = H_I(t)U_I(t, 0). \quad (4)$$

where

$$H_I(t) = e^{\frac{i}{\hbar} \int_0^t H_0(t') dt'} V e^{-\frac{i}{\hbar} \int_0^t H_0(t') dt'}. \quad (5)$$

It can be verified, the matrix element of  $H_I(t)$  is

$$H_I(t)_{nm} = V_{n-m} e^{\frac{i}{\hbar} (n-m) \int_0^t \omega(t') dt'}. \quad (6)$$

$H_I(t)$  is a Laurent matrix. See the Appendix A for a definition of Laurent matrix. From the theorem of Laurent matrix in the Appendix A,  $[H_I(t), H_I(t')] = 0$ .

From the scattering theory,

$$U_I(t, 0) = P \exp \left[ -\frac{i}{\hbar} \int_0^t H(t') dt' \right], \quad (7)$$

where  $P$  is the time ordering operator. Since  $[H_I(t), H_I(t')] = 0$ , we can remove the time ordering operator  $P$  here.

$$U_I(t, 0) = \exp \left[ -\frac{i}{\hbar} \int_0^t H_I(t') dt' \right]. \quad (8)$$

The solution of the Maryland model Eq. (1) is expressed as the matrix exponential of a Laurent matrix.

In the paper, we use Eq. (8) to solve two special cases of Maryland model, the original Maryland model [1, 2, 3] and single band Bloch oscillations [4]. As the first Hamiltonian of the form in Eq. (1) was discussed in [1, 2, 3], the model Hamiltonian in Eq. (1) is referred as Maryland model. Another term can be linear Toeplitz system, which emphasizes the matrix structure of the Hamiltonian and the role of the time ordering operator.

## II. CONNECTION WITH BLOCH OSCILLATIONS

The physical explanation of the Hamiltonian in Eq. (1) can be the original Maryland model [1, 2, 3]. In the rotator representation [1, 2, 3],  $H_0 = -i\hbar \frac{\partial}{\partial \theta} \times \omega(t)$ ,  $V = V(\theta, t)$  and  $|n\rangle$  is  $\frac{1}{\sqrt{2\pi}} e^{in\theta}$ . In another physical explanation,  $|n\rangle$  is seen as a site on an one dimensional lattice.  $V$  is treated as the kinetic energy which causes the electron hopping between different sites and  $H_0$  the potential energy of the electron in the linear electric field. The Hamiltonian of the quantum kicked rotator

$$H = -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \theta^2} - k \cos \theta \sum_{n=1}^{\infty} \delta(t - n\tau) \quad (9)$$

is just a lattice in a harmonic potential  $V(x) = \frac{1}{2} \hbar^2 x^2$ . The kick strength  $k$  is the free diffusion time of the electron.

In the field of quantum chaos, the term “dynamic localization” [5, 6] means the absence of diffusion in the momentum space of the quantum kicked rotator when the kick frequency and the rotator frequency is incommensurable. Dunlap *et al* also used the term “dynamic localization” in the Bloch oscillations problem (See Section V) [4]. The result of the localization of the original Maryland model (See Section IV) is unexpected from the perspective of quantum chaos. But it is just Bloch oscillations. We use the term Bloch oscillations in a broader sense, while the ordinary meaning is an lattice electron in a time independent linear electric field. Even if the

hopping matrix elements between different sites are time dependent, under some condition, the electron still oscillates on the lattice. The localization mechanisms of quantum kicked rotator [6] and the original Maryland model and single band Bloch oscillations are fundamentally different.

### III. EXACT SOLUTIONS IN THE ROTATOR REPRESENTATION

In Section I, we solve the Maryland model in the site representation, now we solve it in the rotator representation. The Hamiltonian of Eq. (1) in the rotator representation is

$$H = -i\omega(t)\hbar\frac{\partial}{\partial\theta} + V(\theta, t), \quad (10)$$

where the unperturbed Hamiltonian

$$H_0 = -i\omega(t)\hbar\frac{\partial}{\partial\theta} = \omega(t)p, \quad (11)$$

where the angular momentum operator  $p = -i\hbar\frac{\partial}{\partial\theta}$ . Since

$$\exp\left\{-\frac{i}{\hbar}\int_0^t \omega(t') dt' p\right\} \quad (12)$$

is a translation operator, in the interaction picture,

$$\begin{aligned} H_I(t) &= e^{\frac{i}{\hbar}\int_0^t \omega(t') dt' p} V(\theta, t) e^{-\frac{i}{\hbar}\int_0^t \omega(t') dt' p} \\ &= V\left(\theta + \int_0^t \omega(t') dt', t\right). \end{aligned} \quad (13)$$

$H_I(t)$  is a function of position  $\theta$ , so  $[H_I(t), H_I(t')] = 0$ . So the time ordering operator can also be removed here and

$$U_I(t, 0) = \exp\left\{-\frac{i}{\hbar}\int_0^t V\left[\theta + \int_0^{t'} \omega(t'') dt'', t'\right] dt'\right\}. \quad (14)$$

The unitary operator in Schrodinger picture is

$$\begin{aligned} U_S(t, 0) &= e^{\frac{i}{\hbar}\int_0^t H_0 dt} U_I(t, 0) \\ &= e^{-\frac{i}{\hbar}\int_0^t \omega(t') dt' p} \times \\ &\quad \exp\left\{-\frac{i}{\hbar}\int_0^t V\left[\theta + \int_0^{t'} \omega(t'') dt'', t'\right] dt'\right\}. \end{aligned} \quad (15)$$

Since

$$\begin{aligned} (H_I(t))_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} V\left(\theta + \int_0^t \omega(t') dt', t\right) e^{i(m-n)\theta} d\theta \\ &= V_{nm} e^{i(n-m)(\int_0^t \omega(t') dt')}, \end{aligned} \quad (16)$$

where  $V_{nm} = V_{n-m} = \frac{1}{2\pi} \int_0^{2\pi} V(\theta, t) e^{i(m-n)\theta} d\theta$ . The equivalence between the methods in Section I and this Section can be easily verified.

In a  $M$  dimensional space, the solution of the Maryland model

$$H = \sum_{m=1}^M \omega_m(t) p_m + V(x_1, x_2, \dots, x_M, t) \quad (17)$$

is

$$\begin{aligned} U(t, 0) &= e^{-\frac{i}{\hbar}\int_0^t \sum_{m=1}^M \omega_m(t') p_m dt'} \times \\ &\quad \exp\left\{-\frac{i}{\hbar}\int_0^t V\left[x_1 + \int_0^{t'} \omega_1(t'') dt'', x_2 + \int_0^{t'} \omega_2(t'') dt'', \right. \right. \\ &\quad \left. \left. \dots, x_M + \int_0^{t'} \omega_M(t'') dt'', t'\right] dt'\right\}, \end{aligned} \quad (18)$$

where  $x_m$  is the position and the momentum  $p_m = -i\hbar\frac{\partial}{\partial x_m}$ .

A special case of Eq. (17) and (18) is Luttinger model [7, 8]

$$H = \sum_{n=1}^N p_{1,n} + \sum_{m=1}^M -p_{2,m} + \sum_{n=1}^N \sum_{m=1}^M V(x_{1,n} - x_{2,m}). \quad (19)$$

$$\begin{aligned} U(t, 0) &= \exp\left\{-\frac{i}{\hbar}\left(\sum_{n=1}^N p_{1,n} + \sum_{m=1}^M -p_{2,m}\right)t\right\} \times \\ &\quad \exp\left\{-\frac{i}{\hbar}\left(\sum_{n=1}^N \sum_{m=1}^M V(x_{1,n} - x_{2,m})\right)t\right\}, \end{aligned} \quad (20)$$

where  $x_{1,n}$ ,  $p_{1,n} = -i\hbar\frac{\partial}{\partial x_{1,n}}$  are position and momentum of the  $n$ -th of “1” particles (electrons) and  $x_{2,n}$ ,  $p_{2,n} = -i\hbar\frac{\partial}{\partial x_{2,n}}$  are position and momentum of the  $n$ -th of “2” particles (holes) and  $N$  and  $M$  are the total number of “1” and “2” particles respectively. The linear Toeplitz structure of Hamiltonian in Eq. (19) may be the origin of anomalous properties of Luttinger model and Luttinger liquid compared with Fermi liquid [7, 8, 9].

### IV. THE ORIGINAL MARYLAND MODEL

The Hamiltonian of the original Maryland model or quantum kicked linear rotator (QKLR) is

$$H = -i\hbar\frac{\partial}{\partial\theta} + V(\theta) \sum_{n=1}^{\infty} \delta(t - n\tau), \quad (21)$$

where  $\theta$  is the angle of the rotator,  $\tau$  the kick period and  $k$  the kick strength.  $p = -i\hbar\frac{\partial}{\partial\theta}$  is the angular momentum and  $V(\theta) = k \cos \theta$  [1, 2, 3]. QKLR is unphysical because there is not a ground state. The classical version of Eq. (21) is not chaotic. The phase space is filled with invariant curves [3]. Does QKLR delocalize if  $\frac{\tau}{2\pi}$  is rational? Berry proposed the energy of the rotator grows

quadratically [3]. From the perspective of the exactly solved eigenstates of the Floquet operator [1, 2, 6], the extended eigenstates generally mean delocalization. But we will prove in the section, QKLR always localizes except one case when  $\tau$  is an integer multiple of  $2\pi$ . We set  $\hbar = 1$  and restrict  $\tau$  to the domain  $(0, 2\pi)$ . The Floquet operator

$$F = e^{-iV(\theta)} e^{-ip\tau}. \quad (22)$$

The matrix representation of the QKLR Hamiltonian is

$$H = \begin{pmatrix} \cdots & \cdots & & & \\ \cdots & -2 & \frac{k}{2}f & & \\ & \frac{k}{2}f & -1 & \frac{k}{2}f & \\ & \frac{k}{2}f & 0 & \frac{k}{2}f & \\ & & \frac{k}{2}f & 1 & \frac{k}{2}f \\ & & & \frac{k}{2}f & 2 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}, \quad (23)$$

where  $f = f(t) = \sum_{n=1}^{\infty} \delta(t - n\tau)$ . In the interaction picture,

$$H_I(t) = \begin{pmatrix} \cdots & \cdots & & & \\ \cdots & 0 & \frac{k}{2}e^{-it}f & & \\ & \frac{k}{2}e^{it}f & 0 & \frac{k}{2}e^{-it}f & \\ & \frac{k}{2}e^{it}f & 0 & \frac{k}{2}e^{-it}f & \\ & & \frac{k}{2}e^{it}f & 0 & \cdots \\ & & & \cdots & \cdots \end{pmatrix}. \quad (24)$$

$H_I(t)$  is a Laurent matrix.  $[H_I(t_1), H_I(t_2)] = 0$ . From Eq. (8), to calculate  $U_I(t, 0) = \exp[-i \int_0^t H_I(t') dt']$ , we need to calculate  $\int_0^t H_I(t') dt'$ . When  $t = N\tau$ , the only non-zero matrix elements of  $\int_0^{N\tau} H(t') dt'$  are

$$\begin{aligned} & \left( \int_0^{N\tau} H(t') dt' \right)_{n+1, n} \\ &= \frac{k}{2} \frac{e^{i\tau}(1 - e^{iN\tau})}{1 - e^{i\tau}} = \frac{k}{2} \frac{\sin(\frac{N\tau}{2})}{\sin(\frac{\tau}{2})} e^{i(N+1)\frac{\tau}{2}} = \gamma e^{i\delta}, \end{aligned} \quad (26)$$

where  $\gamma = \frac{k}{2} \frac{\sin(N\tau/2)}{\sin(\tau/2)}$  and  $\delta = (N+1)\tau/2$ , and

$$\left( \int_0^{N\tau} H(t') dt' \right)_{n, n+1} = \gamma e^{-i\delta}. \quad (27)$$

Since  $\gamma \leq \frac{k}{2\sin(\tau/2)}$ ,  $U(t, 0)$  is a (almost) band matrix. The rotator localizes, whether  $\frac{\tau}{2\pi}$  is rational or irrational except when  $\tau$  is an integer multiple of  $2\pi$ . But the localization mechanism is different from quantum kicked rotator [6] as we discussed in Section II.

From the Appendix B,

$$U(N\tau, 0)_{Inm} = e^{-i(m-n)\delta} i^{m-n} J_{n-m}(2\gamma). \quad (28)$$

In Schrodinger picture,

$$U(N\tau, 0)_{Snm} = e^{-inN\tau} e^{-i(m-n)\delta} i^{m-n} J_{n-m}(2\gamma). \quad (29)$$

If  $\frac{\tau}{2\pi} = \frac{p}{q}$ , the period of QKLR is  $q$  kicks ( $2\pi p$ ). In quantum kicked rotator,  $F = \exp(-ik \cos \theta) \exp(-ip^2 \frac{\tau}{2})$ . When  $\tau = 2\pi$ ,  $F$  is the same with QKLR with  $\tau = \pi$ . So the period of quantum kicked rotator with  $\tau = 2\pi$  is  $2\tau$  ( $4\pi$ ) [5].

Now we discuss the seemingly conflict between the unitary operator  $F^N$  and eigenstates of the Floquet operator  $F$  calculated in [1, 2, 3]. We calculate eigenvalues and eigenstates using a method similar to [3]. The eigenvalue equation of the Floquet operator  $F$  is

$$F\phi(\theta) = e^{-iV(\theta)} e^{-ip\tau} \phi(\theta) = \lambda \phi(\theta). \quad (30)$$

Since  $e^{-ip\tau}$  is a translation operator,

$$e^{-iV(\theta)} e^{-ip\tau} \phi(\theta) = e^{-iV(\theta)} \phi(\theta - \tau) = \lambda \phi(\theta). \quad (31)$$

We assume  $\frac{\tau}{2\pi} = \frac{p}{q}$ . From Eq. (31),

$$\begin{aligned} e^{-iV(\theta-\tau)} \phi(\theta - 2\tau) &= \lambda \phi(\theta - \tau); \\ e^{-iV(\theta-2\tau)} \phi(\theta - 3\tau) &= \lambda \phi(\theta - 2\tau); \\ &\vdots \\ e^{-iV(\theta-(q-1)\tau)} \phi(\theta - q\tau) &= \lambda \phi(\theta - (q-1)\tau). \end{aligned} \quad (32)$$

From Eq. (31), (32) and  $\phi(\theta - q\tau) = \phi(\theta)$ ,

$$\exp(-i \sum_{n=0}^{q-1} V(\theta - n\tau)) = \lambda^q. \quad (33)$$

$$\begin{aligned} \sum_{n=0}^{q-1} V(\theta - n\tau) &= \sum_{n=0}^{q-1} k \cos(\theta - n\tau) \\ &= \cos(\theta + (q-1)\frac{\tau}{2}) \frac{\sin(\frac{q\tau}{2})}{\sin(\frac{\tau}{2})} \\ &= 0. \end{aligned} \quad (34)$$

Another derivation of Eq. (34) is

$$\begin{aligned} \sum_{n=0}^{q-1} k \cos(\theta - n\tau) &= k \operatorname{Re} \left[ \sum_{n=0}^{q-1} \exp(i(\theta - n\tau)) \right] \\ &= k \operatorname{Re} \left[ \exp(i\theta) \sum_{n=0}^{q-1} \exp(i(-n\tau)) \right] \\ &= 0, \end{aligned} \quad (35)$$

where  $\operatorname{Re}$  is the real part of a complex number. So

$$\lambda^q = 1. \quad (36)$$

$$\lambda = 1, e^{i\frac{2\pi}{q}}, e^{i2\frac{2\pi}{q}}, \dots, e^{i(q-1)\frac{2\pi}{q}}. \quad (37)$$

The Floquet operator  $F$  has only  $q$  eigenvalues.  $F$  for rational  $\frac{\tau}{2\pi}$  has infinite degenerate point spectra. Grempel

*et al* thought quasienergy bands have a finite width [1]. Now we find eigenstates. Let's restrict  $\theta$  to the domain  $[0, \frac{2\pi}{q}]$ . Define  $\phi(\theta)$  in the domain  $[0, \frac{2\pi}{q}]$  as an arbitrary function and in other domains

$$\begin{aligned}\phi(\theta - \tau) &= \lambda e^{iV(\theta)} \phi(\theta); \\ \phi(\theta - 2\tau) &= \lambda e^{iV(\theta - \tau)} \phi(\theta - \tau); \\ &\vdots \\ \phi(\theta - (q-1)\tau) &= \lambda e^{iV(\theta - (q-2)\tau)} \phi(\theta - (q-2)\tau).\end{aligned}\tag{38}$$

Eq. (38) gives the eigenstates of  $F$ .

When  $\frac{\tau}{2\pi}$  is irrational, the eigenvalues are  $e^{il\tau}$ , where  $l$  is an arbitrary integer, and the eigenstates are localized [1, 2, 3]. From Eq. (38), we can construct localized and extended eigenstates. There are no real conflict between the unitary operator in Eq. (28) and (29) and the extended eigenstates calculated in [1, 2, 3].

## V. TIME DEPENDENT $H_0$

The time dependent  $H_0$  considered by Dunlap *et al* [4] is

$$\begin{aligned}H(t) &= T \sum_{m=-\infty}^{\infty} (|m\rangle\langle m+1| + |m+1\rangle\langle m|) \\ &\quad + E(t) \sum_{m=-\infty}^{\infty} m|m\rangle\langle m|,\end{aligned}\tag{39}$$

where  $T$  is the nearest-neighbor coupling and  $E(t)$  is the time dependent linear electric potential. Dunlap *et al* gave the analytic solution of the above Hamiltonian [4] and found the time dependent field  $H_0$  generally destroys Bloch oscillations.. Here we treat Eq. (39) as a Maryland model or a linear Toeplitz system.

$$\begin{aligned}H(t) &= \sum_n \sum_{m=-\infty}^{\infty} T_n(t) |m+n\rangle\langle m| \\ &\quad + E(t) \sum_{m=-\infty}^{\infty} m|m\rangle\langle m|.\end{aligned}\tag{40}$$

In the interaction picture,

$$H_I(t) = \sum_n \sum_{m=-\infty}^{\infty} T_n(t) e^{in \int_0^t E(t') dt'} |m+n\rangle\langle m|. \tag{41}$$

$H_I(t)$  is a Laurent matrix.  $U(t, 0)_I = \exp[-i \int_0^t H_I(t') dt']$ . In the simple case of [4],  $T_1(t) = T_{-1}(t) = T$  and  $E(t) = -E \sin t$ . In  $N$  periods from 0 to  $2N\pi$ , for  $n = 1$ ,

$$\begin{aligned}&\int_0^{2N\pi} T_1(t') e^{i \int_0^{t'} E(t'') dt''} dt' \\ &= T \int_0^{2N\pi} e^{i \int_0^{t'} -E \sin t'' dt''} dt' = 2\pi NT J_0(E) e^{-iE},\end{aligned}\tag{42}$$

and for  $n = -1$ ,

$$\int_0^{2N\pi} T_{-1}(t') e^{-i \int_0^{t'} E(t'') dt''} dt' = 2\pi NT J_0(E) e^{iE}.\tag{43}$$

In the derivation of Eq. (42) and (43), we used the formula

$$e^{iz \cos \theta} = \sum_{n=-\infty}^{\infty} J_n(z) i^n e^{in\theta}, \tag{44}$$

of [10, 11]. From the Appendix B, the matrix elements of the unitary operator in the interaction picture is

$$U(t, 0)_{Inm} = e^{i(m-n)E} i^{m-n} J_{n-m}(4\pi NT J_0(E)). \tag{45}$$

Since

$$e^{i \int_0^{2\pi N} H_0(t') dt'} = 1, \tag{46}$$

the unitary operator in Schrodinger picture

$$U(2N\pi, 0)_S = U(2N\pi, 0)_I. \tag{47}$$

From Eq. (46) and (47), the electron will diffuse away except when  $J_0(E) = 0$ . In the case of  $J_0(E) = 0$ , the electron will not delocalize. This is referred as dynamic localization by Dunlap *et al* [4].

## VI. CONCLUSION AND DISCUSSION

In summary, the Maryland model with doubly infinite Hilbert space is exactly solved. In the interaction representation, the unitary operator is the matrix exponential of a Laurent matrix. It is the special structure renders the Hamiltonian solvable. We think the solution can be generalized to a more general structure of Hamiltonian. We give the correct solution of the original Maryland model, concerning the resonant cases. Compared with Dunlap *et al*'s method [4] to solve the lattice electron in a time dependent linear electric field, our method based on the Maryland model is physically appealing and simpler. Further work should generalize the structure of linear Toeplitz system and remove the requirement of doubly infinite Hilbert space.

## APPENDIX A: A THEOREM OF LAURENT MATRIX

"Doubly infinite dimensional Toeplitz matrix" is referred as Laurent matrix in the mathematical literature. Two Laurent matrices commute. In the appendix, we give a proof (it may exist in another place) of the commutativity. A Laurent matrix  $A$  is defined as  $A_{nm} = A_{n+i, m+i} = A_{n-m}$ , where  $i$  is an arbitrary integer and  $n, m$  run from  $-\infty$  to  $\infty$ .  $C = AB$ .  $C_{nm} = \sum_l A_{nl} B_{lm}$ .  $D = BA$ .  $D_{nm} = \sum_p B_{np} A_{pm}$ .

$$\begin{aligned}
C_{nm} - D_{nm} &= \sum_l A_{nl} B_{lm} - \sum_p B_{np} A_{pm} \\
&= \sum_l A_{n-l} B_{l-m} - \sum_p B_{n-p} A_{p-m} \quad (\text{A1}) \\
&= \sum_l A_{n-l} B_{l-m} - \sum_p A_{p-m} B_{n-p}
\end{aligned}$$

Given a  $l$ , there is a  $p = n + m - l$ , which satisfies  $n - l = p - m$  and  $l - m = n - p$ , and *vice versa*. So  $\sum_l A_{n-l} B_{l-m}$  and  $\sum_p A_{p-m} B_{n-p}$  contain the same terms.  $C_{nm} - D_{nm} = 0$ .  $C = D$ .  $[A, B] = 0$ . Note the condition of doubly infinite dimension is necessary to ensure  $p = n + m - l$  is always the index of an existent matrix element.

In fact, the above theorem is trivial because every Laurent matrix  $A$  is a function of the position  $\theta$  (a multiplication operator) in the position representation.

$$A(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{-in\theta}. \quad (\text{A2})$$

Two functions of position commute with each other.

## APPENDIX B: MATRIX EXPONENTIAL OF A LAURENT MATRIX

We now calculate the matrix exponential  $e^{-iM}$  of a simple Laurent Hermitian matrix  $M$ .  $M$  is bidiagonal;

only  $M_{n+1,n}$  and  $M_{n,n+1}$  are not zero.

$$\begin{aligned}
M_{n+1,n} &= \gamma e^{i\delta}; \\
M_{n,n+1} &= \gamma e^{-i\delta},
\end{aligned} \quad (\text{B2})$$

where  $\gamma$  and  $\delta$  are real numbers.

If we choose the basis of  $M$  as  $\frac{1}{\sqrt{2\pi}} e^{in\theta}$ , in the rotator representation,

$$M(\theta) = \gamma e^{i\delta} e^{i\theta} + \gamma e^{-i\delta} e^{-i\theta} = 2\gamma \cos(\theta + \delta). \quad (\text{B3})$$

The matrix element of  $e^{-iM}$  is

$$\begin{aligned}
(e^{-iM})_{nm} &= \frac{1}{2\pi} \int_0^{2\pi} e^{-iM(\theta)} e^{i(m-n)\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-i2\gamma \cos(\theta+\delta)} e^{i(m-n)\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-i2\gamma \cos(\theta+\delta)} e^{i(m-n)(\theta+\delta)} e^{-i(m-n)\delta} d\theta \quad (\text{B4}) \\
&= e^{-i(m-n)\delta} i^{n-m} J_{n-m}(-2\gamma) \\
&= e^{-i(m-n)\delta} i^{n-m} (-1)^{n-m} J_{n-m}(2\gamma) \\
&= e^{-i(m-n)\delta} i^{m-n} J_{n-m}(2\gamma),
\end{aligned}$$

where  $J$  is the Bessel function of the first kind. In the derivation of Eq. (B4), we used Eq. (44) and

$$J_n(z) = (-1)^n J_n(-z) \quad (\text{B5})$$

of [10, 11].

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